Attribution Analysis: Issues Old and New

Abstract

This paper describes in detail the problem of attribution analysis. We suggest a precise approach of solving this problem. The approach is universal and specifically it solves the problem of attribution analysis of return on fund level as well as on sector level. The main shortcomings of other suggested approaches to Attribution Analysis are considered. We show that despite claims to the contrary they are imprecise. A methodology is suggested to find the determinant of portfolio return. Based on the example of historical data for Australian investment funds, we have found that Security Selection is the determinant of return.

Introduction

In the previous article (Kirievsky L. and A. Kirievsky - KK) we described our approach to Attribution Analysis. The current article was contemplated prior to that publication, but for a number of reasons it was postponed for five years[1].

Originally we planned to express our point of view only about the approach to the problem of finding a factor, determining fund returns, raised in a well-known article of Brinson, Hood and Beebower (BHB) and discussed in a number of subsequent publications. We believe the solution of this problem is based on Attribution Analysis (AA) approach as it is defined below.

Since (KK) some new schemes have been proposed to conduct Attribution Analysis of compound returns under multi-period time interval, somewhat similar to the Frank/Russell approach analysed in (KK). In our opinion it is of interest to compare these methods with those discussed previously. To present our analysis and results we have to formulate the problem of AA and the suggested approach in greater detail, than it was done in KK. Consequently, the article is divided into three parts:

- Our formulation of the problem and the suggested approach with examples
- The discussion of BHB methodology from the point of view of AA and the methodology of finding the portfolio return determinant.
- The analysis of published procedures for AA in multi-period time frame.

I. THE PROBLEM OF ATTRIBUTION ANALYSIS AND THE METHOD TO SOLVE IT

The term Attribution Analysis[2] is widely known not only in the finance industry, but outside as well. An Internet search for AA leads to such hits as the AA of Northern Hemisphere mean temperature in dependencies of solar, gas and volcanic forcing, AA of Computer Self-efficacy and many others.

Approaches used to attribute the return of a fund and the analysis of the relationship between the atmospheric temperature and solar, gas and volcanic forces are fundamentally different. In the former, a function is evident in the mathematical sense defined on some set of feasible values of analysed factors. In the latter a statistical hypothesis with a non-evident function based on some selected factors is being tested. Thus, we begin by formulating what we see as a problem of AA[3].

To begin, we describe what is given:

- There is a set of one or more factors, for example u, v, w (readers, who do not like the term "factor" can substitute the term "variable" instead[4]).
- A set Ω of feasible values for all factors is given.
- There is a procedure, which produces a number corresponding to a particular set of factor values. Thereby a function f is defined (in our case it is $f(u,v,w)$), which is called the analysing or the objective function.
- Two points in the set Ω are selected, which are sets $\{u_1,v_1,w_1\}$ and $\{u_2,v_2,w_2\}$, of feasible values for the factors. One is called the benchmark position and the second one is called the analysing position. The corresponding values for the objective function $f(u_1,v_1,w_1)$ and $f(u_2,v_2,w_2)$ are computed.

Now, let's formulate what we are trying to find:

- The problem lies in attributing a share of the difference $\Delta f = f(u_2,v_2,w_2) - f(u_1,v_1,w_1)$ between the values of the function to the changes in the value of each factor Δf_u, Δf_v, Δf_w from the benchmark position to the analysed position separately from changes in all other factors (that is when other factors are equal to their values in the benchmark position). We will refer to Δf_u, Δf_v and Δf_w as pure effects from changes in factors ***u***, ***v*** and ***w*** correspondingly.

Such a formulation of the problem is natural when analysing a function of a number of variables. Let us note that the change in function as a result of a change in values for the variables is determined by the difference in the values of the function and not their ratio, and this difference decomposes into the sum of components, not their product. This takes place whilst no assumptions are made as to how the underlying function is calculated. It can be "geometric" in character (e.g. product of $(1+R_t)$ terms over time), "non geometric" "calibrated" risk adjusted return $(\sigma_M/\sigma)(R-R_f) + R_f$ (Modigliani et al) or even given in a non-analytical form, such as a procedural approach found in the Morningstar risk-adjusted rating (Sharpe).

It follows that the sum of pure effects $\Delta f_u,+\Delta f_v,+\Delta f_w$ can be different from Δf. The difference ***I*** between them must be attributed to simultaneous changes in more than one factor. This is often referred to as ***Interaction*** or ***Cross Product***.

From the above formulation of the problem an obvious solution follows (KK):

$$\Delta f_u = f(u_2,v_1,w_1) - f(u_1,v_1,w_1)$$
$$\Delta f_v = f(u_1,v_2,w_1) - f(u_1,v_1,w_1)$$
$$\Delta f_w = f(u_1,v_1,w_2) - f(u_1,v_1,w_1)$$
$$I = \Delta f - (\Delta f_u,+\Delta f_v,+\Delta f_w)$$

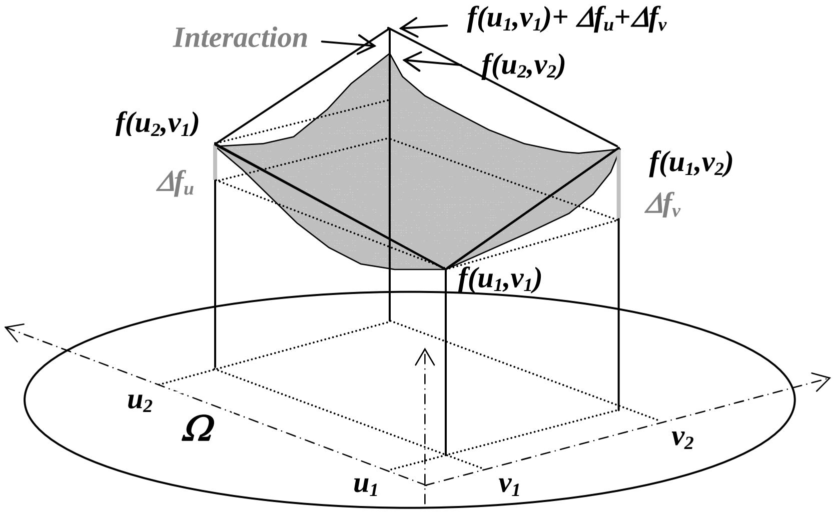
The approach is used when all intermediate points are feasible, that is (u_2,v_1,w_1), (u_1,v_2,w_1) and (u_1,v_1,w_2) belong to Ω.

There is a simple and understandable geometric interpretation of the AA problem[5]:

- Let there be two factors ***u*** and ***v*** . They define a plane, to which the Ω belongs
- In Ω the points (u_1,v_1) and (u_2,v_2) are given. The points (u_1,v_2) and (u_2,v_1) also belong to Ω
- The value of function ***f*** is presented on the vertical axis.

The plane, drawn through points $(u_1,v_1, f(u_1,v_1))$, $(u_1,v_2, f(u_1,v_2))$ and $(u_2,v_1, f(u_2,v_1))$, crosses the vertical line, plotted through a point $(u_2,v_2,0)$, in a point $f(u_1,v_1)+ \Delta f_u,+\Delta f_v$, which differs from $f(u_2,v_2)$. The difference between these two values is Interaction.

Figure 1



From the definition of Interaction there is no objective procedure for its distribution among factors. Nevertheless some subjective approaches for further decomposition of this remainder into components, attributable to changes in individual factors have been suggested (including KK). We believe that in each such instance it is necessary to fully describe the process of decomposition and the exact function to which it is applied. That is, for which factors, for which objective function it is applied, what remainder and how is decomposed.

Hereafter we will consider sectors weights and returns as factors and the compound return as an objective function $\boldsymbol{F}$. As mentioned above, the suggested approach would not change if, for example, Modigiani's risk adjusted return was used as an objective function instead. However the standard compound return is more convenient for the purposes of analysis of approaches in other publications, some of which are only applicable to compound returns.

Let's say we have n sectors and we analyse the compound return, obtained over a time interval, which comprises of T periods. The function F is defined on a set of values of weights $\boldsymbol{w_{it}}$ and returns $\boldsymbol{r_{it}}$:

$$F = \prod_{t=1}^{T}(1+\sum_{i=1}^{n} w_{it} r_{it}) - 1$$

The set $\boldsymbol{\Omega}$ of feasible factors values is defined by the constraints[6]:

$$\sum_{i=1}^{n} w_{it} = 1 \text{ for any } t; \quad w_{it} \geq 0 \text{ for some } i \in [1..n] \text{ and } t \in [1..T] \qquad (1)$$

When considering the aggregate values of $\{w_{it}\}$ and $\{r_{it}\}$ for all $i \in [1..n]$ and $t \in [1..T]$ as the two factors, we obtain the problem of Attribution Analysis of the compound return on portfolio level. When the values of $\{w_{it}\}$ and $\{r_{it}\}$ are further separated into subsets for each value of the index $\boldsymbol{i}$ we obtain the problem on a sector level[7].

By selecting different points in the feasible set of values of returns and weights different variants of the problem of attribution analysis can be obtained:

1. Let's select as the benchmark the weights from the Strategic Asset Allocation (SAA, or Policy Allocation in BHB) of the fund and indices returns of the Asset Sectors (passive returns in BHB) and the actual returns and weights of the fund as the point of analysis. Considering the set of weights and the set of returns as two factors we obtain the problem of attributing Tactical Asset Allocation (TAA) and Security Selection (SS) decisions.
2. Let's consider the peer group, to which this fund belongs. If the SAA of the funds within the group differ, then we can analyse the consequences of the deviation of one fund's SAA from the SAA of another, which can act as the benchmark. We consider it to be more fruitful to determine the strategy, common to all members of the peer group and to subsequently attribute the consequences of deviations from this common strategy to the SAA of a particular fund. As the common strategy one can select the average strategy of all the group members or calculate the weighted average strategy, with weights dependent upon the size of the members or their style. Selecting the SAA of the benchmark fund or the common strategy of the group and indices returns of the Asset Sectors as the benchmark and as the point of analysis the benchmark from Problem 1 above we end up with a single factor problem of evaluating the result of deviation of the fund's strategy relative to the benchmark.
3. Let's consider as the Attribution Analysis benchmark some fixed SAA, such as 100% in cash or the average strategy of a number of peer groups, and as a point of analysis the benchmark from 2 above. Once again we end up with a single factor problem of evaluating the result of deviation of the fund's or peer group strategy relative to the selected fixed strategy.
4. Let's select the same benchmark as in 1 above and the combination of benchmark weights and real sector returns as our analysed point. It is evident that with such a choice sector returns $\{r_{it}\}$ vary. It is natural to select as n factors $\{r_{it}, t \in [1..T]\}$. This produces the problem of SS on sector level.

Evidently, the abovementioned problems can be combined. For example the difference between returns of the analysed fund in 1 and benchmark in 2 can be attributed to three factors SAA, TAA and SS, or to only two - the full Asset Allocation and SS[8].

To demonstrate the application of the suggested methodology let's introduce the following designations:

A series of values (vector) $\{a_1, a_2, \ldots a_T\}$ is presented as $\vec{a}$.
SAA weights are presented as $\{\vec{w}_{bi}, i \in [1..n]\}$, index returns as $\{\vec{r}_{bi}, i \in [1..n]\}$,
actual sector returns as $\{\vec{r}_{pi}, i \in [1..n]\}$ and actual sector weights as $\{\vec{w}_{pi}, i \in [1..n]\}$.

Then, the return of the benchmark is given by[9]:

$$F_{bb} = F(\vec{w}_{bi}, \vec{r}_{bi}) = \prod_{t=1}^{T}(1 + \sum_{i=1}^{n} w_{bit} r_{bit}) - 1$$

Similarly, the return of the fund is given by:

$$F_{pp} = F(\vec{w}_{pi}, \vec{r}_{pi}) = \prod_{t=1}^{T}(1 + \sum_{i=1}^{n} w_{pit} r_{pit}) - 1$$

Applying the methodology to Problem 1, by varying sector weights we obtain the TAA effect. By varying sector returns we obtain the SS effect:

$$TAA = F_{pb} - F_{bb} = \prod_{t=1}^{T}(1 + \sum_{i=1}^{n} w_{pit} r_{bit}) - \prod_{t=1}^{T}(1 + \sum_{i=1}^{n} w_{bit} r_{bit})$$

$$SS = F_{bp} - F_{bb} = \prod_{t=1}^{T}(1 + \sum_{i=1}^{n} w_{bit} r_{pit}) - \prod_{t=1}^{T}(1 + \sum_{i=1}^{n} w_{bit} r_{bit})$$

Consequently:

$$Interaction = (F_{pp} - F_{bb}) - (F_{pb} - F_{bb}) - (F_{bp} - F_{bb})$$

Note that "points" $(\{\vec{w}_{pi}, i \in [1..n]\}, \{\vec{r}_{bi}, i \in [1..n]\})$ and $(\{\vec{w}_{bi}, i \in [1..n]\}, \{\vec{r}_{pi}, i \in [1..n]\})$ are feasible. In other words, when for each $t \in [1..T]$ all weights w_{it} are changed simultaneously such that the constraints (1) are satisfied, it is possible to calculate return in the new point and therefore the approach can be applied.

We believe that in this problem it is inexpedient to decompose Interaction between TAA and SS[10].

The solution for single factor problems 2 and 3 is obvious: changes in the objective function are fully attributed to changes of the single factor.

To solve Problem 4 let's define the following points in the set Ω:

$$P_i = (\{\vec{w}_{bi}, i \in [1..n]\}, \{\vec{r}_{bj}, j \in [1..n], j \neq i, \vec{r}_{pi}\})$$

From the definition it follows that each point P_i is different from the point, defining the benchmark only by the vector of returns of the *i* sector: the weights and returns of all sectors, except *i* are the same as of the benchmark. We calculate the value of *F* in these points and call them $F_{b(b/pi)}$. Then, pure effects of changes in sector returns SS_i are equal to the differences between $F_{b(b/pi)}$ and F_{bb}, and the remainder of subtracting SS_i from *SS*, defined in the problem 1 is the cross-product from the concurrent changes of more than one sector return:

$$SS_i = F_{b(b/pi)} - F_{bb}$$

$$I_{SS} = F_{bp} - F_{bb} - \sum_{i=1}^{n}(F_{b(b/pi)} - F_{bb})$$

In KK an approach was suggested for further decomposition of the cross product for the problem of AA on the sector level[11]. The idea of the approach lies in "equal attitude" to each sector. In addition to the pure effect, each sector receives half of the residual effect, which this sector participated in creating. The procedure is simple in terms of calculations and produces insignificant net error. Let's illustrate its application on the decomposition of I_{SS}:

Similarly to the points P_i let's define points Q_i in the set Ω:

$$Q_i = (\{\vec{w}_{bi}, i \in [1..n]\}, \{\vec{r}_{pj}, j \in [1..n], j \neq i, \vec{r}_{bi}\})$$

Each point Q_i is different from the analysing point $(\{\vec{w}_{bi}, i \in [1..n]\}, \{\vec{r}_{pi}, i \in [1..n]\})$, which defines the full SS effect, by the value of the return of the *i* sector. Let's designate a value of the objective function in Q_i as $F_{b(p/bi)}$.

Excess return $F_{b(p/bi)}$ - F_{bb} contains the full effect of changes in all sector returns, except for *i*. Therefore, the difference F_{bp}- $F_{b(p/bi)}$ contains the pure effect SS_i and all other effects from changes in return of the *i* sector. Half of these effects (that is the part of I_{SS}, which is attributed to the return of *i* sector) is equal to 0.5(F_{bp}- $F_{b(p/bi)}$ - SS_i), and the part of *SS*, attributed to *i* sector with the part of the cross product I_{SS}, is calculated according to the formula:

$$\underline{SS_i} = 0.5(F_{bp} - F_{b(p/bi)} - SS_i) + SS_i = 0.5(F_{bp} - F_{b(p/bi)} + SS_i)$$
$$= 0.5[(F_{bp} - F_{b(p/bi)}) + (F_{b(b/pi)} - F_{bb})]$$

The next example illustrates the suggested approach. Table 1 presents the data on portfolio and benchmark weights and returns for the three sectors and three periods. Table 2 presents results of intermediate calculations and the analysis of attribution on the fund and sector levels. For Problem 4, Table 2 contains pure effects SS_1, SS_2, SS_3 and their sums with corresponding parts of distributed cross-product $\underline{SS_1}$, $\underline{SS_2}$, $\underline{SS_3}$, respectively.

Table 1

		Period		
Portfolio		1	2	3
Equities	w	50.0	60.0	60.0
	r	5.0	4.0	3.8
Fixed Interest	w	40.0	30.0	20.0
	r	2.5	2.0	2.4
Cash	w	10.0	10.0	20.0
	r	2.0	2.4	2.2
Benchmark				
Equities	w	40.0	40.0	40.0
	r	4.0	4.0	4.0
Fixed Interest	w	40.0	40.0	40.0
	r	3.0	3.0	3.0
Cash	w	20.0	20.0	20.0
	r	2.0	2.0	2.0

Table 2

	Single Period			**Multiple Periods**
	1	2	3	
F_{pp}	3.70	3.24	3.20	10.485796
F_{bb}	3.20	3.20	3.20	9.910477
F_{bp}	3.40	2.88	2.92	9.484155
F_{pb}	3.40	3.50	3.40	10.657646
SS = F_{bp} - F_{bb}	0.20	-0.32	-0.28	-0.426322
TAA = F_{pb} - F_{bb}	0.20	0.30	0.20	0.747169
Interaction	0.10	0.06	0.08	0.254472
$F_{b(b/p1)}=F_{b(p1b2b3)}$	3.60	3.20	3.12	10.250954
$F_{b(b/p2)}=F_{b(b1p2b3)}$	3.00	2.80	2.96	9.018166
$F_{b(b/p3)}=F_{b(b1b2p3)}$	3.20	3.28	3.24	10.038313
$SS_1=F_{b(b/p1)} - F_{bb}$	0.40	0.00	-0.08	0.340477
$SS_2=F_{b(b/p2)} - F_{bb}$	-0.20	-0.40	-0.24	-0.892310
$SS_3=F_{b(b/p3)} - F_{bb}$	0.00	0.08	0.04	0.127836
$I_{SS}=(F_{bp} - F_{bb}) - (SS_1+SS_2+SS_3)$	0.00	0.00	0.00	-0.002324
$F_{b(p/b1)} = F_{b(b1p2p3)}$	3.00	2.88	3.00	9.145392
$F_{b(p/b2)} = F_{b(p1b2p3)}$	3.60	3.28	3.16	10.379219
$F_{b(p/b3)} = F_{b(p1p2b3)}$	3.40	2.80	2.88	9.356502
$\langle SS\rangle_1=F_{bp} - F_{b(p/b1)}$	0.40	0.00	-0.08	0.338763
$\langle SS\rangle_2=F_{bp} - F_{b(p/b2)}$	-0.20	-0.40	-0.24	-0.895064
$\langle SS\rangle_3=F_{bp} - F_{b(p/b3)}$	0.00	0.08	0.04	0.127654
$\underline{SS_1}=(SS_1+\langle SS\rangle_1)/2$	0.40	0.00	-0.08	0.339620
$\underline{SS_2}=(SS_2+\langle SS\rangle_2)/2$	-0.20	-0.40	-0.24	-0.893687
$\underline{SS_3}=(SS_3+\langle SS\rangle_3)/2$	0.00	0.08	0.04	0.127745
Round$=(F_{bp} - F_{bb}) - (\underline{SS_1}+\underline{SS_2}+\underline{SS_3})$	0.00	0.00	0.00	0.000001

II. ATTRIBUTION ANALYSIS AND DETERMINANTS OF PORTFOLIO PERFORMANCE

In 1986 Brinson, Hood and Beebower published the framework for Attribution Analysis of funds returns and the results of their calculations, according to which the "asset allocation policy contributes more than 90% to the return performance of large pension funds" (BHB). The methodology for the BHB calculations was repeated in (Brinson et al, 1991). Virtually simultaneously with this publication an article (Hensel, Don Ezra and Ilkiw) appeared in the same journal critiquing of the suggested methodology.

In the mid 90s a new wave of criticisms appeared (Surz et al, Jahnke), which generally repeated the observations of C.Hensel et al. Finally, in 2000 Ibbotson and Kaplan formulated three questions "about the importance of asset allocation", defining the scope of discussion:

1) "How much of the variability of returns across time is explained by policy (SAA decision)?"
2) "How much of the variation in returns among funds is explained by differences in policy?"
3) "What portion of the return level is explained by policy return?"(Ibbotson et al).

Our view on BHB, formed after reading criticism changed significantly after a detailed analysis of the 1986 article. Below we expound our opinion of BHB.

Firstly, let's define the objectives pursued by Brinson, Hood and Beebower. In our opinion that can be derived from the article's name: "what factors determine actual returns of investment funds". Consequently, to the questions posed by Ibbotson and Kaplan we can add a fourth:

What factor (investment decision: strategic asset allocation, tactical asset allocation or security selection) determines funds returns better than others?

To obtain its objectives authors selected a universe of 91 large pension plans with 10 years of quarterly sector allocations and returns.For every fund they tried to solve two Attribution Analysis problems:

- Attributing changes in a fund's compound return to Tactical Asset Allocation (TAA) and Security Selection (SS) decisions, and
- Evaluating results of deviation in a fund's strategy relative to the benchmark (problems 1 and 2 from the list above).

To solve Problem 1 the authors used the same calculation scheme, as we described 10 years later. In solving the single-factor Problem 2 the authors deviated from this methodology. They did not define a benchmark and instead of attributing changes in strategic return to SAA, they attributed the entire strategic return[12,13].

Having conducted Attribution Analysis for all funds from the universe, authors produce average values (-0.66, -0.36, 10.11 for TAA, SS and strategic return respectively and –0.07 for Interaction), standard deviation (0.49, 1.36 and 0.22 for TAA, SS and strategic return and 1.45 for active return), as well as maximum and minimum values. From the fact that obtained average figures for strategic return are significantly larger than for TAA and SS (which are in fact negative!), authors reach the conclusion about the "ability of investment policy to dictate actual plan [fund] return".

Further the authors attempt to quantitatively measure the influence of a fund's strategic return on its actual return. With this the attribution values obtained for the selected interval are not used: the problem of AA are solved for every single quarter in that 10 years interval and a regression of time series of actual return and strategic return ran. As was noted by R.Ibbotson and P.Kaplan, such calculations answer the question of "how much of fund's variability in return through time could be explained by variations in strategic return (93.6%)", which differs from the BHB's objectives, as we understand them.

The mistake of BHB in substituting changes in strategic return with strategic return itself, was noted as far back as Hensel et al. To rectify the problem they suggested using one of the following benchmarks:

1. 100% allocation in cash
2. 100% allocation in bonds
3. average allocation of the funds from a peer group[14]
4. market mix (asset allocation used in a market index)

Note that in any case authors suggest that the same benchmark be chosen in analysing strategic asset allocation of all funds in the universe.

The choice of a benchmark significantly affects values attributed to the SAA decision, and correspondingly the analysed ratios of SAA to SS, TAA, active return and total return itself. However, as with any choice of the benchmark, the same value of F_{bb} is used in calculating changes in strategic return for all funds. Therefore the relative position of SAA does not change. To determine this position (ranking SAA) strategic return can be used instead of SAA (note that the standard deviation of SAA also does not change due to the choice of the benchmark).

The following methodology can be introduced to answer the question "Which factor better determines the return of the fund?":

1. For all funds in a peer group we calculate the ranking of compound return R_{Fi}, TAA and SS attributing values R_{TAAi} and R_{SSi} and strategic return R_{SAAi}.
2. Calculate the sum of deviations of R_{Fi} from ranking of each of the factors:
   - $\sum_i Abs(R_{Fi} - R_{TAAi})$
   - $\sum_i Abs(R_{Fi} - R_{SSi})$
   - $\sum_i Abs(R_{Fi} - R_{SAAi})$

The factor, for which this sum is the smallest, better determines the relative position of a fund's return in its peer group than others. Complementarily we can calculate the rank correlation coefficients (Spearman rank coefficients) $r = 1 - \frac{6\sum_i D_i^2}{n(n^2-1)}$, where $\boldsymbol{D_i}$ is a rank deviation for a $\boldsymbol{i}$-th fund and $\boldsymbol{n}$ is a number of funds. Attribution value ranks of the factor with bigger $\boldsymbol{r}$ better correlate with ranks of the total returns.

We applied the described methodology to the results of attribution analysis for 26 Australian growth funds for 1997-1999 and 2000-2002 (Tables 3, 4 and 5) [15]. Furthermore, we added active return to the list of factors as well as the information about ranking deviations:

- the per cent of funds with deviation, which is less than 3 and 5
- the maximum deviation for each factor and rankings of all factors for the fund, for which the maximum was obtained

As expected those maximum deviations occur when factor rankings are found to be at the different ends of the potential spectrum of rating values, that is when the fund performs really well in one area, but poorly in another area (R_{TAA} = 25 and R_{SS} = 1, R_{SAA} = 5 and R_{SS} = 22, R_{TAA} = 3 and R_{SS} = 25 and so on).

Table 3: Rank Deviations of Growth Funds Attributions in Period 1997-1999

	Sum of Deviat.	Deviat. per Fund	% of Deviat. <=3	% of Deviat. <=5	Spearman coefficient	Maximum Deviation in Ranks					
						Max Value	Rank R	Rank SAA	Rank TAA	Rank SS	Rank AR
SAA	158	6.077	30.80%	50.00%	0.48	15	9	24	21	2	5
TAA	210	8.077	23.10%	30.80%	0.14	24	1	4	25	1	1
SS	96	3.692	50.00%	61.50%	0.77	13	11	1	15	24	24
Active Return	84	3.231	57.70%	76.90%	0.80	13	11	1	15	24	24

Table 4: Rank Deviations of Growth Funds Attributions in Period 2000-2002

	Sum of Deviat.	Deviat. per Fund	% of Deviat. <=3	% of Deviat. <=5	Spearman coefficient	Maximum Deviation in Ranks					
						Max Value	Rank R	Rank SAA	Rank TAA	Rank SS	Rank AR
SAA	171	6.577	23.10%	42.30%	0.41	17	22	5	25	22	24
TAA	172	6.615	26.90%	38.50%	0.43	15	16	26	1	2	2
SS	102	3.923	34.60%	73.10%	0.76	14	16	26	1	2	2
Active Return	86	3.308	53.80%	73.10%	0.80	14	16	26	1	2	2

Table 5: Rank Deviations of Growth Funds Attributions in Period 1997-2002

	Sum of Deviat.	Deviat. per Fund	% of Deviat. <=3	% of Deviat. <=5	Spearman coefficient	Maximum Deviation in Ranks					
						Max Value	Rank R	Rank SAA	Rank TAA	Rank SS	Rank AR
SAA	194	7.462	26.90%	38.50%	0.20	22	23	1	19	26	26
TAA	196	7.538	26.90%	38.50%	0.13	23	26	24	3	25	25
SS	80	3.077	46.20%	84.60%	0.83	15	25	21	26	10	23
Active Return	68	2.615	69.20%	80.80%	0.88	12	16	26	1	13	4

As can be seen from Tables 3, 4, and 5, SS is better able than SAA and TAA to determine the relative position of the fund's return, and therefore it is the determinant of return.

III. OTHER PROPOSALS ON HOW TO "COMBINE ATTRIBUTION EFFECTS OVER TIME"

In a number of publications on the topic of Attribution Analysis of return for the multi-period time interval authors formulate principles or laws which the calculation procedure must follow. By a lucky chance their approaches always satisfy the criteria, whilst alternative approaches do not.

We believe that the only thing an approach must do is to solve the problem it was designed to address. Thus, it is better to talk about method's properties, rather than principles it should follow.

Two of the properties that are found in a large number of publications are "intuitiveness" and "precision (exactness, accuracy, no residual etc)".

It is easier to deal with intuitiveness. If completely different procedures for AA can all be claimed by their authors to be intuitive, it means that intuitiveness is a highly subjective property, where the decision on whether a method is intuitive or not belongs to the user [16].

It becomes more difficult with the term "precision". Most authors understand "precision" to mean that the difference between the change in the value of the function (Δf) and the sum of components to which such value is decomposed is equal to zero. In other words, if for a given Attribution Analysis procedure such difference does not equal zero, the procedure would be imprecise. However, then by adding the non-zero difference to any of the components, such procedure would suddenly become precise!

In the commonly considered case, where the fund return is the objective function, sector weights $\{w_{it}\}$ and returns $\{r_{it}\}$ are factors and strategic return acts as a benchmark, it can be formulated as follows:

> Excess return has to be exactly equal to the sum of attributes for tactical asset allocation, security selection and interaction. If it is not equal, then it is sufficient to change any or all of the components so that equality would be restored, and the property of "precision" would be realised.

Hence, instead of calculating attribution for a multi-period time interval one can, for example, calculate attributes for each single period, add up the results for $\boldsymbol{TAA_t}$, $\boldsymbol{SS_t}$ and $\boldsymbol{I_t}$, and then adjust the three values in a way mentioned above.

The authors of such methods claim their proposed approach to be "without residuals", neglecting the fact that Interaction is exactly the residue of excess return, stripped of pure effects of factors.

In regards to this we would like to quote Craig Wainscott from Frank Russell (FR) Company (Wainscott):

"The interaction effect is a residual when the allocation and selection effects are calculated in their purest forms. …
The results [compounded allocation, selection and interaction effects] … represent a trade-off between complete precision and ease of understanding. Compounding the attribution effects generates precise results but does not produce totals that are simply additions of the period-by-period effects. FR has found that explaining performance attribution to clients is easier if the effects add up than if they are absolutely precise. In the FR approach, therefore, the effects are calculated on a monthly basis and compounded; then, the differences are rounded off and smoothed so that the effects appear to be additive again."

It can be said that the only three principles, which the methodology should follow were described above in formulating the Attribution Analysis Problem:

1. Accurate description of factors $\boldsymbol{u, v, w\ldots}$, a set of feasible values $\boldsymbol{\Omega}$ and an objective function $\boldsymbol{f(u,v,w\ldots)}$.
2. A precise computation of pure effects $\boldsymbol{\Delta f_u}$, $\boldsymbol{\Delta f_v}$, $\boldsymbol{\Delta f_w}$...
3. A detailed description of the procedure for dealing with the cross product
   $\boldsymbol{I = f(u_2,v_2,w_2\ldots) - f(u_1,v_1,w_1\ldots) - (\Delta f_u,+\Delta f_v,+\Delta f_w+\ldots)}$

In the last three years a number of schemes were proposed, which do not comply with such principles. Consequently, they can be described as imprecise, approximate or incorrect, depending on the preferred term. Next, let's consider the particulars of some of those schemes and the claimed advantages.

In (Menchero Fall 2000) a scheme was introduced for Attribution Analysis of return under the multi-period time interval, similar to the Frank Russell scheme described in (Carino) and analysed in (KK). The difference between the value of the actual return R_{pp} and the benchmark R_{bb} for the entire time period is given as the weighted sum of single period differences $R_{ppt} - R_{bbt}$, which are decomposed into separate effects applying the standard formulae. The difference lies in formulae for the weights used in summation. In FR methodology they are equal to the ratio k_t/k , where
$k_t = (ln(1+ R_{ppt}) - ln(1+ R_{bbt}))/(R_{ppt} - R_{bbt})$, and $k = (ln(1+ R_{pp}) - ln(1+ R_{bb}))/(R_{pp} - R_{bb})$.
In (Menchero Fall 2000) they are given as $\boldsymbol{A} + \boldsymbol{\alpha_t}$, where $\boldsymbol{\alpha_t}$'s are proportionate to $(R_{ppt} - R_{bbt})$. In the analysis of the compound return the FR approach appears to us more intuitive: at small x $ln(1+x) \sim x$, therefore, k_t is close to 1, and formulas for k_t and k logically follow from the formula for the compound return.

Both schemes suffer from similar shortcomings, namely 1) the application of coefficients dependent upon all factors for calculation of a pure effect of every factor (so, resulting effects are not pure!), 2) the impossibility of extracting of pure effects on the sector level, and 3) the difference in approach in analysis on the portfolio level and on the sector level. We would also believe the following requires a comment. In (Menchero Fall 2000) the scheme is referred to as being "optimised" and in the body of the article it is highlighted that coefficients $\boldsymbol{\alpha_t}$ "optimally distribute the residual among the different time periods". Formally $\boldsymbol{\alpha_t}$'s are obtained as a solution of an extremum problem, so thus they can be referred to as "optimised". However, this extremum problem possesses an artificial character: $\boldsymbol{\alpha_t}$'s can be calculated from the condition of their proportionality to $(R_{ppt} - R_{bbt})$ without an optimisation process as such[17]. As we see it, regardless of the presence or absence of the process of optimisation, the scheme for calculating $\boldsymbol{\alpha_t}$'s does not define the character of Attribution Analysis approach and therefore it is too ambitious to call this approach "optimised". The author seems to have recognised this and proposed a geometric approach to Attribution Analysis in a subsequent publication (Menchero Winter 2000/2001).

Geometric approach to Attribution Analysis of return is not new. As far as 13 years ago in a blueprint for a standard proposed in the Q-group (Australia), in February 1991, a geometric approach was recommended and a corresponding scheme given. This idea was not developed further primarily because it was much more difficult to interpret than the widely accepted arithmetic scheme.

The developers of the geometric approach proceed from the presumption that with the geometric character of the total return (more accurately, the function $F=1+R$) its depiction, as a product of effects from factors will be simpler and more understandable, than as a summation of effects. The difficulties in interpreting the results of the geometric decomposition in case of a single period time interval prove the opposite. Another significant argument against the use of the geometric approach follows from the fact that whilst using this approach the effects of individual sectors are multiplied, but the sector returns in calculating the portfolio return are added. In the next example we apply the (Menchero Winter 2000/2001) approach to the simplest case of a single period time interval and only two sectors. We obtain the following decomposition[18]:

$$\frac{1+R_{pp}}{1+R_{bb}} = \frac{1+w_{p1}r_{p1}+w_{p2}r_{p2}}{1+w_{b1}r_{b1}+w_{b2}r_{b2}} = (1+SS_1)*(1+SS_2)*(1+TAA_1)*(1+TAA_2) =$$

$$\left(\frac{1+w_{p1}r_{p1}}{1+w_{p1}r_{b1}}*\Gamma^{SS}\right)*\left(\frac{1+w_{p2}r_{p2}}{1+w_{p2}r_{b2}}*\Gamma^{SS}\right)*\left(\frac{1+w_{p1}r_{b1}}{1+w_{b1}r_{b1}}*\frac{1+w_{b1}R_b}{1+w_{p1}R_b}*\Gamma^{TAA}\right)*\left(\frac{1+w_{p2}r_{b2}}{1+w_{b2}r_{b2}}*\frac{1+w_{b2}R_b}{1+w_{p2}R_b}*\Gamma^{TAA}\right),$$

where

$$\Gamma^{SS} = \left(\frac{1+w_{p1}r_{p1}+w_{p2}r_{p2}}{1+w_{b1}r_{b1}+w_{b2}r_{b2}}*\frac{1+w_{p1}r_{b1}}{1+w_{p1}r_{p1}}*\frac{1+w_{p2}r_{b2}}{1+w_{p2}r_{p2}}\right)^{0.5},$$

$$\Gamma^{TAA} = \left(\frac{1+w_{p1}r_{p1}+w_{p2}r_{p2}}{1+w_{b1}r_{b1}+w_{b2}r_{b2}}*\frac{1+w_{b1}r_{b1}}{1+w_{p1}r_{b1}}*\frac{1+w_{p1}R_b}{1+w_{b1}R_b}*\frac{1+w_{b2}r_{b2}}{1+w_{p2}r_{b2}}*\frac{1+w_{p2}R_b}{1+w_{b2}R_b}\right)^{0.5}$$

Compare it with the arithmetic approach:

$$R_p - R_b = SS_1 + SS_2 + TAA_1 + TAA_2, \text{ where}$$
$$SS_1 = w_{b1}*(r_{p1}-r_{b1});\quad SS_2 = w_{b2}*(r_{p2}-r_{b2});$$
$$TAA_1 = (w_{p1}-w_{b1})*(r_{b1}-R_b);\quad TAA_2 = (w_{p2}-w_{b2})*(r_{b2}-R_b);$$

If we take the pairs $\{w_1, r_1\}$ and $\{w_2, r_2\}$ as the two factors, then the geometric approach results in the following sector attribution:

$$\frac{1+w_{p1}r_{p1}+w_{p2}r_{p2}}{1+w_{b1}r_{b1}+w_{b2}r_{b2}} = \left(\frac{1+w_{p1}r_{p1}}{1+w_{b1}r_{b1}}*\frac{1+w_{b1}R_b}{1+w_{p1}R_b}\right)*\left(\frac{1+w_{p2}r_{p2}}{1+w_{b2}r_{b2}}*\frac{1+w_{b2}R_b}{1+w_{p2}R_b}\right)*\Gamma,$$

where

$$\Gamma = \frac{1+w_{p1}r_{p1}+w_{p2}r_{p2}}{1+w_{b1}r_{b1}+w_{b2}r_{b2}}*\frac{1+w_{b1}r_{b1}}{1+w_{p1}r_{p1}}*\frac{1+w_{p1}R_b}{1+w_{b1}R_b}*\frac{1+w_{b2}r_{b2}}{1+w_{p2}r_{p2}}*\frac{1+w_{p2}R_b}{1+w_{b2}R_b}$$

whilst the arithmetic scheme gives:

$$(w_{p1}r_{p1}+w_{p2}r_{p2})-(w_{b1}r_{b1}+w_{b2}r_{b2}) =$$
$$[w_{p1}r_{p1}-w_{b1}r_{b1}-(w_{p1}-w_{b1})*R_b]+[w_{p2}r_{p2}-w_{b2}r_{b2}-(w_{p2}-w_{b2})*R_b]$$

In our opinion this geometric approach can hardly be described as “highly intuitive and mathematically sound”.

Bacon is a proponent of the geometric approach. At the beginning of his article (Bacon) he makes an evident statement that the ratio of excess return for the full time interval to the return of the benchmark at time T is not equal to the ratio of the same excess return to the return of the benchmark at time T-1:

$$\frac{R_{pp}(T)-R_{bb}(T)}{R_{bb}(T-1)} \neq \frac{R_{pp}(T)-R_{bb}(T)}{R_{bb}(T)}$$

from which he concludes that the geometric approach is preferable (?).

Bacon claims that he attributes a "geometric excess return" function $\frac{R_{pp}-R_{bb}}{1+R_{bb}}$. In reality he uses $F=1+R$ as an objective function and a decomposition, which was firstly suggested in early 1990s $\frac{F_{pp}}{F_{bb}}=\frac{F_{pp}}{F_{pb}}\cdot\frac{F_{pb}}{F_{bb}}$. The value $\frac{F_{pp}}{F_{pb}}-1$ is declared as the *SS* attribute, and $\frac{F_{pb}}{F_{bb}}-1$ as the *TAA* attribute.

Immediately a question arises, how this approach is better than $\frac{F_{pp}}{F_{bb}}=\frac{F_{pp}}{F_{bp}}\cdot\frac{F_{bp}}{F_{bb}}=(TAA+1)\cdot(SS+1)$. Both approaches are non-symmetrical. A clear analogy can be drawn to the arithmetic approach, where Interaction is included in one of the attributes. It was possible to use a symmetric approach $\frac{F_{pp}}{F_{bb}}=\frac{F_{pb}}{F_{bb}}\cdot\frac{F_{bp}}{F_{bb}}\cdot\frac{F_{pp}F_{bb}}{F_{pb}F_{bp}}=(TAA+1)\cdot(SS+1)\cdot\frac{F_{pp}F_{bb}}{F_{pb}F_{bp}}$, however how do we then interpret the cross-product?

The next problem of the suggested approach lies in interpreting the resulting expression for the "geometric excess return":

$$\frac{R_{pp}-R_{bb}}{1+R_{bb}}=\frac{F_{pp}}{F_{bb}}-1=(TAA+1)\cdot(SS+1)-1=TAA\cdot SS+TAA+SS\ .$$

Finally, how can we apply this approach in solving the AA problem on sector level?

In (Mirabelli) another approach for Attribution Analysis of a compound return was proposed, which differs somewhat from those arithmetic schemes considered earlier. The author believes that this approach combines the benefits of both geometric and arithmetic approaches. As we mentioned above, in (Carino) and (Menchero Fall 2000) the difference between the value of the actual return R_{pp} and the benchmark R_{bb} during the entire time period is presented as a weighted sum of single period differences $R_{ppt} - R_{bbt}$. In the (Mirabelli) approach the change in the actual return during the entire time interval is also given as the sum of single period differences, but modernised differences are used, namely instead of $R_{ppt} - R_{bbt}$ the author uses $R_{ppt}\prod_{\tau=1}^{t-1}(1+R_{pp\tau})-R_{bbt}\prod_{\tau=1}^{t-1}(1+R_{bb\tau})$. Such modernization allows avoiding the use of correction coefficients (weights), but creates a new problem: how to attribute such modernised single period changes of return. In (Mirabelli) modernised coefficients are "hidden" in returns and then a standard decomposition is applied:

$$R_{ppt}\prod_{\tau=1}^{t-1}(1+R_{pp\tau})-R_{bbt}\prod_{\tau=1}^{t-1}(1+R_{bb\tau})=\sum_{i}^{N}w_{pit}r_{pit}\prod_{\tau=1}^{t-1}(1+R_{pp\tau})-\sum_{i}^{N}w_{bit}r_{bit}\prod_{\tau=1}^{t-1}(1+R_{bb\tau})$$
$$=\sum_{i}^{N}w_{pit}\tilde{r}_{pit}-\sum_{i}^{N}w_{bit}\tilde{r}_{bit}=$$
$$\sum_{i}^{N}(w_{pit}-w_{bit})(\tilde{r}_{pit}-\sum_{i}^{N}w_{bit}\tilde{r}_{bit})+\sum_{i}^{N}w_{bit}(\tilde{r}_{pit}-\tilde{r}_{bit})+\sum_{i}^{N}(w_{pit}-w_{bit})(\tilde{r}_{pit}-\tilde{r}_{bit})$$

where $\tilde{r}_{pit}=r_{pit}\prod_{\tau=1}^{t-1}(1+R_{pp\tau})$ and $\tilde{r}_{bit}=r_{bit}\prod_{\tau=1}^{t-1}(1+R_{bb\tau})$

Such approach suffers from the same shortcomings as the (Carino) and (Menchero Fall 2000) approaches. However the author highlights a specific property of this approach as being an advantage: its "incremental" character, which allows one to estimate the effectiveness of the change in the factor within each separate time period[19]. In our opinion this should not be taken into consideration at all. An optimal value for the factor in each separate time period often leads to global inefficiency of selected values over the entire multi period time interval. As was noted in [7], we believe such approach to be methodologically incorrect.

In (Frongello) a scheme is suggested to modernize single period differences, which is similar to the approach in (Mirabelli). The author begins by stating that "single period methodologies today not only differ in regards to which attributes to present but also in regards to *how* these attributes are presented", after which he selects a following approach for single periods:

$$R_{ppt} - R_{bbt} = SS + TAA; \quad SS = R_{ppt} - R_{pbt}; \quad TAA = R_{pbt} - R_{bbt}$$

The choice of such a non-symmetric approach with adding Interaction to *SS* is inconsequential to the following discussion but is nevertheless of interest.

Next, the author lists the characteristics, which an Attribution Analysis method should follow. To the three characteristics, taken from (Carino), being Generality, Familiarity and No Residuals Frongello adds another three: Sincerity, Intuitive and Order Dependence.

Order Dependence is present in the Mirabelli approach. It means that differences $R_{ppt} - R_{bbt}$ are modernised with the use of weight coefficients, that depend on returns in previous time periods.. We do not see why in developing an Attribution Analysis method one should think about whether the results depend on the order of single periods.

The suggested in Frongello scheme is not new. We found it described in AIPMPS:

$$\begin{aligned} R_{pp} - R_{bb} &= (1+R_{ppt})\prod_{\tau=1}^{t-1}(1+R_{pp\tau}) - (1+R_{bb\tau})\prod_{\tau=1}^{t-1}(1+R_{bb\tau}) \\ &= (1+R_{ppt}-1-R_{bbt})\prod_{\tau=1}^{t-1}(1+R_{pp\tau}) + (1+R_{bbt})\prod_{\tau=1}^{t-1}(1+R_{pp\tau}) - (1+R_{bbt})\prod_{\tau=1}^{t-1}(1+R_{bb\tau}) \\ &= (R_{ppt}-R_{bbt})\prod_{\tau=1}^{t-1}(1+R_{pp\tau}) + (1+R_{bb\tau})\left[\prod_{\tau=1}^{t-1}(1+R_{pp\tau}) - \prod_{\tau=1}^{t-1}(1+R_{bb\tau})\right] \end{aligned}$$

As a result we obtain a recurrent relation: the attribute from period $[1..T]$ is equal to the attribute from period $[1..T-1]$, multiplied by $(1+R_{bbt})$, plus the attribute for the last simple period T, multiplied by the fund return for period $[1..T-1]$, plus 1. Frongello states that this approach is intuitive and understandable to "individuals without highly mathematical background".

The property of "Sincerity" is formulated as follows: "The model should be devoid of any mathematical fudging used in order to satisfy any of the desirable characteristics". We believe that to use the actual fund return $\prod_{\tau=1}^{T-1}(1+R_{pp\tau})$ to find the value of the attribute for the period $[1..T]$, that is a value, which depends on changes in all factors, to calculate the effect of changes in just one factor, is an example of such mathematical fudging, which should be avoided.

The articles by D.Laker (Davies and Laker, 2001, Laker, 2001, Laker, 2002) deserve a special mention. Since becoming acquainted with KK and our presentation Laker actively advocates our approach in journals, conferences and on the Internet. Such an energetic supporter can only be welcomed, provided that such support is done properly [20]. However, in relation to abovementioned publications we would like to make the following comments:

1. We agree with Laker, that the approach "is this simple". However, as we highlighted in our presentations, the approach simply and naturally flows from the formulation of the problem, as shown in KK and here. The presence of different approaches to AA for the multi-period time interval can be explained by the fact that their authors have not attended our presentations and explanations in KK were not clear or detailed enough (however Laker did attend such conferences and was consulted by us on numerous occasions, so his poor understanding of our approach is more puzzling). If a problem is formulated differently, albeit with the same name AA, the solution, generally speaking, would also be different.
2. In their articles Davies and Laker refer to their own unpublished work. One such article is the "unpublished discussion paper circulated in 1997". The second is a working document, which we were able to find in archives (AIPMPS). In that document Cross Products are considered as a result

of "Linking Single Period Results" different from Interaction, and suggested four methods "to resolving Cross Products" (one of which is the Frongello approach), but there is nothing even closely resembling our approach. This shows that Davies and Laker did not envisage the presence of this "simple" approach prior to our presentations. At the same time the subsequent articles by Laker fail to mention that they follow KK directly and only repeat some of the results shown in KK. We consider this to be unethical to say the least.

3. Despite the familiarity with KK, related presentations and personal consultations, Laker does not fully understand the suggested approach. Consequently, he mistakenly claims on a number of occasions that "many of the other methods for calculating attributes over multiple periods have the advantage that they automatically produce sector level results", "approximate methods are still necessary for calculating sector-level attributes" and suggested to "use the scaling method … to obtain sector-level attributes that agree with the exact fund-level attributes". We repeat once more: if the problem corresponds to our definition of AA problem, then it can be solved using the suggested approach. Specifically, this relates to Problem 4 from the first part of this paper. As a result, the value of SS attribute at the fund level, obtained after solving Problem 1, decomposes into pure effects of security selection in each sector and the cross product. However this, obviously, is a cross product, different from that in Problem 1.

CONCLUSION

1. This article describes in detail the problem of AA and a precise method of solving it.
2. It is shown that the suggested approach is universal and specifically it solves the problem of attribution analysis of return on fund level as well as on sector level.
3. The main shortcomings of other suggested approaches to AA are analysed. We show that despite claims to the contrary they are imprecise.
4. A methodology is suggested to find the determinant of portfolio return. Based on the example of historical data for Australian investment funds, we have found that Security Selection is the determinant of return.

REFERENCES

AIPMPS, "Australian Investment Performance Measurement and Presentation Standards", unpublished, April 1997.

Bacon C., "Excess Returns – Arithmetic Or Geometric?", Journal of Performance Measurement, vol.6, no.3, Spring 2002.

Brinson G., L. Hood and G. Beebower, "Determinants of Portfolio Performance", Financial Analysts Journal, July/August 1986.

Brinson G., B. Singer and G. Beebower, "Determinants of Portfolio Performance II: An Update", Financial Analysts Journal, May/June 1991.

Carino D., "Combining Attribution Effects Over Time", Journal of Performance Measurement, vol.3, no.4, Summer 1999.

Chambers English Dictionary, W & R Chambers Ltd, Edinburgh, 1990

Davies, O. and Laker, D. "Multiple-Period Performance Attribution Using the Brinson Model", Journal of Performance Measurement, vol.6, no.1, Fall 2001.

Frongello A., "Linking Single Period Attribution Results", Journal of Performance Measurement, vol.6, no.3, Spring 2002.

Hensel C. D. Don Ezra and J. Ilkiw, "The Importance of the Asset Allocation Decision", Financial Analysts Journal, July/August 1991.

Higgs P. Goode S., “Target active returns and attribution analysis”, Financial Analysts Journal, May/June 1993.

Ibbotson R. and P.Kaplan, “Does Asset Allocation Policy Explain 40, 90, or 100 Percent of Performance?”, Financial Analysts Journal, January/February 2000.

Investment Performance Measurement and Attribution Standard, version 1.3, Q-Group (Australia), February 1991.

Jahnke W., “The Importance of asset allocation”, Journal of Investing, Spring 2000.

Kirievsky L. and A. Kirievsky, “Attribution Analysis: Combining Attribution Effects Over Time Made Easy”, Journal of Performance Measurement, vol.4, no.4, Summer 2000.

Laker, D., “Multiple Period Performance Attribution: A Survey and some Tentative Conclusions”, IIR Performance Measurement & Attribution Conference, Sydney Australia, March 2001

Laker, D., “A View From Down-Under”, ”, Journal of Performance Measurement, vol.6, no.4, Summer 2002.

Los, C., “Optimal multi-currency investment strategies with exact attribution in three Asian countries”, Journal of Multinational Financial Management, Vol. 8, 1998.

Los, C., “Comment on “Combining Attribution Effects Over Time””, Journal of Performance Measurement, vol.4, no.1, Fall 1999.

Menchero J., “An Optimized Approach to Linking Attribution Effects Over Time”, Journal of Performance Measurement, vol.5, no.1, Fall 2000.

Menchero J., “A Fully Geometric Approach to Performance Attribution”, Journal of Performance Measurement, vol.5, no.2, Winter 2000/2001.

Mirabelli A., “The Structure and Visualization of Performance Attribution”, Journal of Performance Measurement, vol.5, no.2, Winter 2000/2001.

Modigliani F., and L.Modigliani “Risk-Adjusted Performance”, Journal of Portfolio Management, Winter 1997.

Sharpe W., “Morningstar’s Risk-Adjusted Ratings”, Financial Analysts Journal, July/August 1998.

Surz J. D. Stevens and M.Wimer, “Investment Policy Explains All”, Journal of Performance Measurement, vol.3, no.4, Summer 1999.

Surz J. D. Stevens and M.Wimer, “The Importance of Investment Policy”, Journal of Investing, Winter 1999.

Wainscott C., “Attribution Analysis for Equities”, Performance Evaluation, Benchmarks, and Attribution Analysis, AIMR 1995

ENDNOTES

[1]We would like to thank the anonymous referee for comments on the August 2001 draft of this article. Those comments and the fact that subsequent publications (Menchero, Mirabelli, Frongello etc.) refer to our approach but do not attempt to compare their suggestions with it, lead us to think that the original description in KK did not provide enough details. We hope to rectify it with the current article.

[2] A number of authors use terms such as Performance Attribution or Return Attribution to highlight the link between their analysis and investment process. However, the name of the term does not guarantee that what is being considered is the problem of AA, as it was formulated in KK and this article. For example C.Los in a letter to the editor of Journal of Performance Measurement (Los, 1999) discusses the need for "simultaneous attribution of return and risk" and contends that he has solved this problem. However he does not formulate or solve the problem of AA in the articles, to which he refers. (Los, 1998). On the other hand, the mathematical formulation of the problem, the methodology of its solution and the scope of its applicability is not unique to the finance industry. This is why we prefer to use the term AA.

[3] In some publications about AA authors express caution that to understand some of the methodologies a deep mathematical background is required and appeal for an approach to be used, which can be understood by investment professionals. We believe that to understand the problem of AA and the approach to its solution all that is required is a good knowledge of high school mathematics.

[4] We use the term "factor" and not "variable" to avoid confusion, as these variables could be scalars, continuous functions of other variables, vector-functions etc.

[5] A number of publications use a diagram, which was employed by Brinson et al (1986) to interpret the suggested approach (Fig. 2). We consider that this diagram, as well as its modification (Fig. 3) is not precise or illustrative enough, and prefer (Fig. 1)

Figure 2

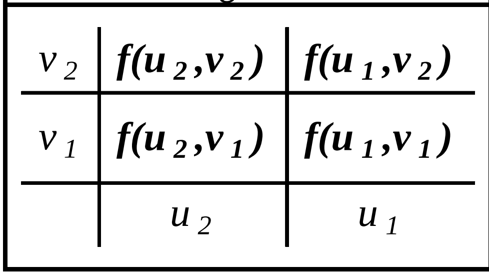
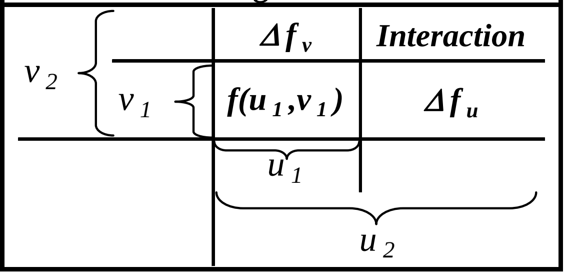


Figure 3



[6] Other constraints on asset allocation are possible, provided, for example by the investors. Their inclusion does not add to the problem conceptually, but complicates the presentation. Thus, such constraints are not included here.

[7] It is possible to formally decompose aggregates $\{w_{it}\}$ and $\{r_{it}\}$ on separate subsets for each time interval t. However we believe such approach to be methodologically flawed: a correct approach is to analyse asset allocation decisions made over a prolonged period of time. It does not exclude the possibility of examining two multi-period intervals, analysing each one and comparing results.

[8] The list specified here can be extended. For example if the data regarding the returns before and after tax is available, then tax can be included as a factor. On the other hand considering the weights and returns of any individual sector as a factor we obtain Attribution Analysis problem on sector level and so on.

[9] As in (KK), the first index of the function $\boldsymbol{F}$ relates to weights, the second to returns (*p* - actual portfolio, *b* – benchmark). If the subscript *t* is absent, then the term refers to vectors of weight and return for all t, otherwise – to a weight and return in the selected period *t*.

[10] A number of authors suggest that Interaction be added to TAA or SS, based on the specifics of investment decision making. We believe that prior to decomposing Interaction in such a way, it is necessary to answer following questions:

1. Does the decision maker, at point of making the current decision, know the future outcome of decisions made earlier?
2. Do investment decisions in some sectors affect future results from other sectors, and if yes, how?

In relation to this we would like to mention the suggestion from (Higgs at al.) to set "target" returns r_{gi} for sectors (the single period case was considered) and calculate

$$TAA = F_{pg} - F_{bg} = \sum_{i=1}^{n} w_{pi} r_{gi} - \sum_{i=1}^{n} w_{bi} r_{gi}$$

$$SS = F_{bg} - F_{bb} = \sum_{i=1}^{n} w_{bi} r_{gi} - \sum_{i=1}^{n} w_{bi} r_{bi}$$

[11] The methodology was developed at the request of users to distribute cross-products from TAA and SS decisions on sector levels and was incorporated into InTech Desktop Consultant software (www.intech.net.au).

[12] We could not find a reference in articles, analysing BHB, to the fact that BHB are calculating attribution for multi-period time intervals. This fact is also absent in articles on AA problem.

However in defence of the authors of those articles (ourselves included) we can note that in BHB and Brinson et al, 1991 in describing the used framework authors do not produce formulae for multi-period attribution, but only for the single period time interval.

[13] In some publications the authors claimed that attributing entire strategic return (F_{pp} from the problem 2) instead of changes in return (F_{pp}-F_{bb}) to SAA is equivalent to using a benchmark of 100% cash allocation. However even in this case the benchmark return is different from 0, that is $F_{bb} \neq 0$.

[14] Hensel et al suggest that a typical constant in time allocation of a peer group be adopted as a benchmark, is 50% US stocks, 5% non US stocks, 30% US bonds, 5% real estate and 10% cash. This is instead of calculating average strategic asset allocation of the funds from a peer group.

[15] Monthly data from a universe of 31 Australian growth funds: strategic and actual asset allocation, actual sector returns and indices returns were supplied by InTech Research Pty Ltd. The funds allocated in 8 sectors: Australian and International Shares, Listed and Unlisted Property, Australian, Australian Inflation Linked and International Bonds and Cash. Out of 31 funds we selected 26, which had data for the entire 1997-2002 period.

[16] If we consider intuition as the "truth of things [real or imaginary] without reasoning or analysis" (Chambers), then many approaches can be considered intuitive. Moreover, when BHB solved Problem 1 (correctly) and Problem 2 (incorrectly), they did so intuitively. We use a similar computation scheme, but based on strict mathematical formulation of the problem, and therefore, it can be said that such approach is not intuitive.

[17] With given $\boldsymbol{A}$, the same $\boldsymbol{\alpha}_t$'s can be calculated without any optimisation as $C*(R_{ppt} - R_{bbt})$, where the constant C is derived from the equation:

$$R_{pp} - R_{bb} = \sum_{t}^{T} \left[A + C(R_{ppt} - R_{bbt})\right](R_{ppt} - R_{bbt}) = A\sum_{t}^{T}(R_{ppt} - R_{bbt}) + C\sum_{t}^{T}(R_{ppt} - R_{bbt})^2$$

Even within the proposed scheme, changes in $\boldsymbol{A}$ lead to other "optimised" coefficients $\boldsymbol{\alpha}_t$.

[18] In the given decomposition, the cross product is added to Security Selection effect as it is done in (Menchero Winter 2000/2001). A further separation of the cross-product makes the decomposition even more complicated.

[19] In the geometric approach it is also often considered as an advantage. We do not consider it to be so. Furthermore, in the Mirabelli case the statement is incorrect, as the correction terms used are themselves dependent upon the values of *all* other factors in other time periods.

[20] At present Laker is advertising our suggested approach under the auspices of Barra, Inc.